Lesson 15. Geometry and Algebra of "Corner Points"

0 Warm up

Example 1. Consider the system of equations

$$\begin{aligned} 3x_1 + x_2 - 7x_3 &= 17 \\ x_1 + 5x_2 &= 1 \\ -2x_1 + 11x_3 &= -24 \end{aligned} \tag{$*$}$$

Let $A = \begin{pmatrix} 3 & 1 & -7 \\ 1 & 5 & 0 \\ -2 & 0 & 11 \end{pmatrix}$. We have that $\det(A) = 84$.

- Does $(*)$ have a unique solution, no solutions, or an infinite number of solutions?

  Unique solution, because $\det(A) \neq 0$

- Are the row vectors of A linearly independent? How about the column vectors of A?

  Yes, the row and column vectors of A are LI, because $\det(A) \neq 0$

- What is the rank of A? Does A have full row rank?

  $\text{rank}(A) = 3$, because $\det(A) \neq 0$ $\Rightarrow$ A has full row rank.

1 Overview

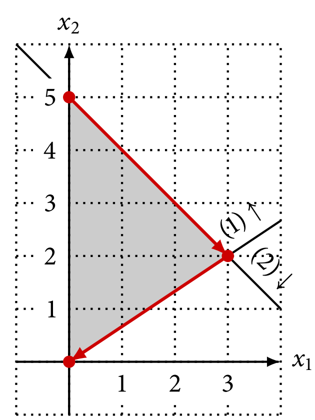
- Due to convexity, local optimal solutions of LPs are global optimal solutions

  $\Rightarrow$ Improving search finds global optimal solutions of LPs

- The simplex method: improving search among "corner points" of the feasible region of an LP

- How can we describe "corner points" of the feasible region of an LP?

- For LPs, is there always an optimal solution that is a "corner point"?



2 Polyhedra and extreme points

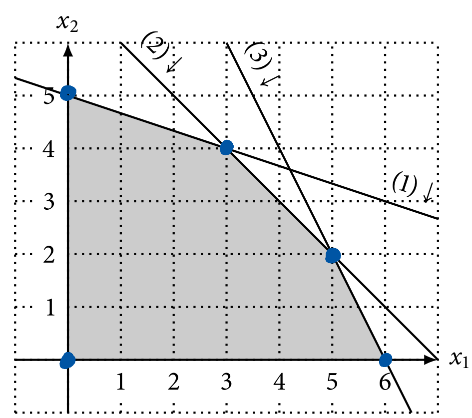
- A **polyhedron** is a set of vectors $\mathbf{x}$ that satisfy a finite collection of linear constraints (equalities and inequalities)
  - Also referred to as a **polyhedral set**
- In particular:

Polyhedron $\Leftrightarrow$ feasible region of LP

- Recall: the feasible region of an LP – a polyhedron – is a convex feasible region
- Given a convex feasible region S, a solution $\mathbf{x} \in S$ is an **extreme point** if there does not exist two distinct solutions $\mathbf{y}, \mathbf{z} \in S$ such that $\mathbf{x}$ is on the line segment joining $\mathbf{y}$ and $\mathbf{z}$
  - i.e. there does not exist $\lambda \in (0,1)$ such that $\mathbf{x} = \lambda\mathbf{y} + (1-\lambda)\mathbf{z}$

Example 2. Consider the polyhedron S and its graph below. What are the extreme points of S?

$$S = \left\{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : \begin{array}{rl} x_1 + 3x_2 \leq 15 & (1) \\ x_1 + x_2 \leq 7 & (2) \\ 2x_1 + x_2 \leq 12 & (3) \\ x_1 \geq 0 & (4) \\ x_2 \geq 0 & (5) \end{array} \right\}$$



- "Corner points" of the feasible region of an LP $\Leftrightarrow$ extreme points

3 Basic solutions

- In Example 2, the polyhedron is described with 2 decision variables
- Each corner point / extreme point is the intersection of 2 lines
- Equivalently, each corner point / extreme point is active at 2 distinct constraints

  constraint is satisfied w/ equality
- Is there a connection between the number of decision variables and the number of active constraints at a corner point / extreme point?
- Convention: all variables are on the LHS of constraints, all constants are on the RHS
- A collection of constraints defining a polyhedron are **linearly independent** if the LHS coefficient matrix of these constraints has full row rank

Example 3. Consider the polyhedron S given in Example 2. Are constraints (1) and (3) linearly independent?

LHS coefficient matrix of (1) + (3): $L = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$

$\det(L) = 1 - 6 = -5 \neq 0 \Rightarrow L$ has full row rank

$\Rightarrow$ (1) + (3) are LI.

$$\begin{aligned} x_1 + 3x_2 &\le 15 & (1)\\ x_1 + x_2 &\le 7 & (2)\\ 2x_1 + x_2 &\le 12 & (3)\\ x_1 &\ge 0 & (4)\\ x_2 &\ge 0 & (5) \end{aligned}$$

- Given a polyhedron S with n decision variables, $\mathbf{x}$ is a **basic solution** if
  (a) it satisfies all equality constraints
  (b) at least n constraints are active at $\mathbf{x}$ and are linearly independent
- $\mathbf{x}$ is a **basic feasible solution (BFS)** if it is a basic solution and satisfies all constraints of S

Example 4. Consider the polyhedron S given in Example 2. Verify that $(3,4)$ and $(21/5, 18/5)$ are basic solutions. Are these also basic feasible solutions?

$n = 2 = $ # decision variables

$$\begin{aligned} x_1 + 3x_2 &\le 15 & (1)\\ x_1 + x_2 &\le 7 & (2)\\ 2x_1 + x_2 &\le 12 & (3)\\ x_1 &\ge 0 & (4)\\ x_2 &\ge 0 & (5) \end{aligned}$$

$(3,4)$: (a) S has no equality constraints $\Rightarrow$ automatically satisfied.

(b) Which constraints are active at $(3,4)$?

(1): $3 + 3(4) = 15$ ✓
(2): $3 + 4 = 7$ ✓
(3): $2(3) + 4 < 12$ ×
(4): $3 > 0$ ×
(5): $4 > 0$ ×

(1)+(2) are active at $(3,4)$

Are (1)+(2) LI?

$L = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$ $\det(L) = 1 - 3 = -2 \neq 0 \Rightarrow$ Yes!

$\Rightarrow$ (b) satisfied

$\Rightarrow (3,4)$ is a basic solution

$(3,4)$ also satisfies all constraints $\Rightarrow (3,4)$ is a BFS.

$\left(\frac{21}{5}, \frac{18}{5}\right)$: (a) S has no equality constraints $\Rightarrow$ automatically satisfied.

(b) Which constraints are active at $\left(\frac{21}{5}, \frac{18}{5}\right)$?

(1): $\frac{21}{5} + 3\left(\frac{18}{5}\right) = 15$ ✓
(2): $\frac{21}{5} + \frac{18}{5} > 7$ ×
(3): $2\left(\frac{21}{5}\right) + \frac{18}{5} = 12$ ✓
(4): $\frac{21}{5} > 0$ ×
(5): $\frac{18}{5} > 0$ ×

(1) + (3) are active at $\left(\frac{21}{5}, \frac{18}{5}\right)$

Are (1)+(3) LI? By Ex. 3, yes!

$\Rightarrow$ (b) satisfied

$\Rightarrow \left(\frac{21}{5}, \frac{18}{5}\right)$ is a basic solution

$\left(\frac{21}{5}, \frac{18}{5}\right)$ is not a BFS, since it violates (2)

$$
\begin{aligned}
x_1 + 3x_2 &\le 15 && (1)\\
x_1 + x_2 &\le 7 && (2)\\
2x_1 + x_2 &\le 12 && (3)\\
x_1 &\ge 0 && (4)\\
x_2 &\ge 0 && (5)
\end{aligned}
$$

Example 5. Consider the polyhedron S given in Example 2.

a. Compute the basic solution $\mathbf{x}$ active at constraints (3) and (5). Is $\mathbf{x}$ a BFS? Why?
b. In words, how would you find all the basic feasible solutions of S?

a. $\left.\begin{aligned} 2x_1 + x_2 &= 12 \\ x_2 &= 0 \end{aligned}\right\} \Rightarrow \vec{x} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$ is basic soln. active at (3)+(5)

$\vec{x}$ satisfies all constraints $\Rightarrow \vec{x}$ is a BFS.

b. Consider all collections of $n=2$ LI constraints.
Solve for the corresponding basic solution, check if feasible

4 Equivalence of extreme points and basic feasible solutions

- From our examples, it appears that for polyhedra, extreme points are the same as basic feasible solutions

Big Theorem 1. Suppose S is a polyhedron. Then $\mathbf{x}$ is an extreme point of S if and only if $\mathbf{x}$ is a basic feasible solution.

- See Rader p. 243 for a proof
- We use "extreme point" and "basic feasible solution" interchangeably

5 Adjacency

- An **edge** of a polyhedron S with n decision variables is the set of solutions in S that are active at $(n-1)$ linearly independent constraints

$$
\begin{aligned}
x_1 + 3x_2 &\le 15 && (1)\\
x_1 + x_2 &\le 7 && (2)\\
2x_1 + x_2 &\le 12 && (3)\\
x_1 &\ge 0 && (4)\\
x_2 &\ge 0 && (5)
\end{aligned}
$$

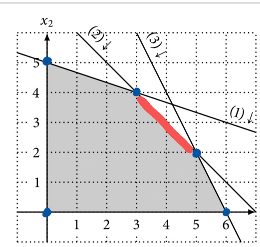
Example 6. Consider the polyhedron S given in Example 2.

a. How many linearly independent constraints need to be active for an edge of this polyhedron?
b. Describe the edge associated with constraint (2).

a. $n = 2 \Rightarrow n-1 = 1$ constraint needs to be active.

b. all solutions in S that satisfy $x_1 + x_2 = 7$

$\Rightarrow$ line segment connecting extreme pts. $(3,4)$ and $(5,2)$

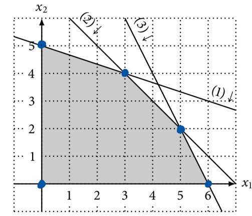


- Edges appear to connect "neighboring" extreme points
- Two extreme points of a polyhedron S with n decision variables are **adjacent** if there are $(n-1)$ common linearly independent constraints at active both extreme points
  - Equivalently, two extreme points are adjacent if the line segment joining them is an edge of S

Example 7. Consider the polyhedron S given in Example 2.

a. Verify that $(3,4)$ and $(5,2)$ are adjacent extreme points.
b. Verify that $(0,5)$ and $(6,0)$ are not adjacent extreme points.

$$\begin{array}{rl} x_1 + 3x_2 \le 15 & (1) \\ x_1 + x_2 \le 7 & (2) \\ 2x_1 + x_2 \le 12 & (3) \\ x_1 \ge 0 & (4) \\ x_2 \ge 0 & (5) \end{array}$$



a. $(3,4)$ is active at $(1)+(2)$, $(5,2)$ is active at $(2)+(3)$ $\Rightarrow$ $(3,4)$ and $(5,2)$ are active at $n-1=1$ common LI constraint

$\Rightarrow$ $(3,4)$ and $(5,2)$ are adjacent.

b. $(0,5)$ is active at $(1)+(4)$, $(6,0)$ is active at $(3)+(5)$ $\Rightarrow$ $(0,5)$ and $(6,0)$ are NOT active at $n-1=1$ common LI constraint

$\Rightarrow$ $(0,5)$ and $(6,0)$ are NOT adjacent

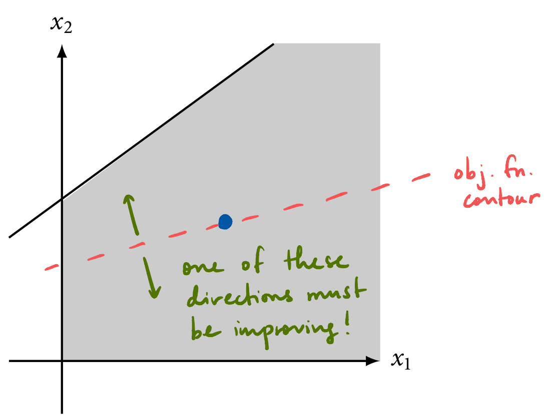
- We can move between adjacent extreme points by "swapping" active linearly independent constraints

6 Extreme points are good enough: the fundamental theorem of linear programming

Big Theorem 2. Let S be a polyhedron with at least 1 extreme point. Consider the LP that maximizes a linear function $\mathbf{c}^\top \mathbf{x}$ over $\mathbf{x} \in S$. Then this LP is unbounded, or attains its optimal value at some extreme point of S.

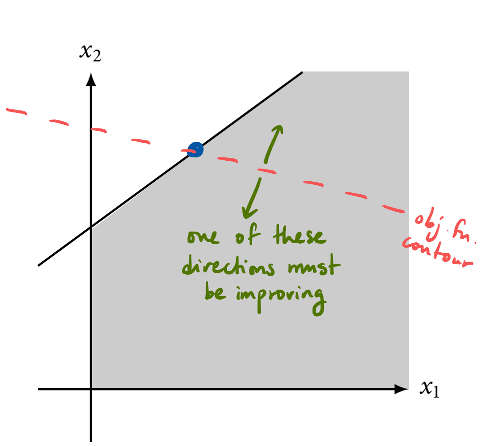
"Proof" by picture.

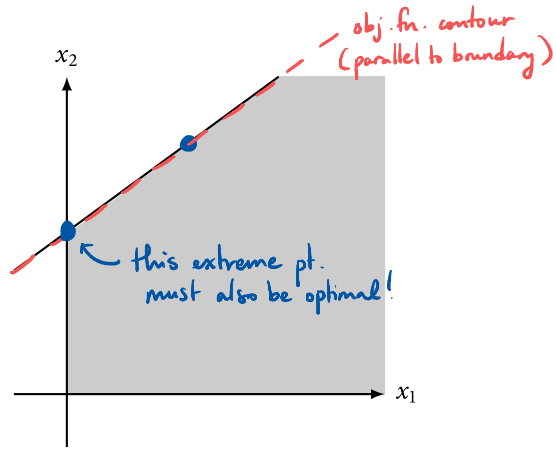
- Assume the LP has finite optimal value
- The optimal value must be attained at the boundary of the polyhedron, otherwise:



$\Rightarrow$ The optimal value is attained at an extreme point or "in the middle of a boundary"

- If the optimal value is attained “in the middle of a boundary”, there must be multiple optimal solutions, including an extreme point:





⇒ The optimal value is always attained at an extreme point □

- **For LPs, we only need to consider extreme points as potential optimal solutions**
- It is still possible for an optimal solution to an LP to not be an extreme point
- If this is the case, there must be another optimal solution that is an extreme point

7 Food for thought

- Does a polyhedron always have an extreme point?
- We need to be a little careful with these conclusions – what if the Big Theorem doesn’t apply?
- Next time: we will learn how to convert any LP into an equivalent LP that has at least 1 extreme point, so we don’t have to be (so) careful